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Non-Equilibrium Steady States of the *XY* Chain

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We study the non-equilibrium statistical mechanics of the two-sided XY chain. We start from an initial state in which the left and right part of the lattice,

$$\mathbb{Z}_{\mathrm{L}} = \{x \in \mathbb{Z} \mid x < -M\}, \qquad \mathbb{Z}_{\mathrm{R}} = \{x \in \mathbb{Z} \mid x > M\},$$

are at inverse temperatures β_{L} and β_{R}. Using a simple scattering theoretic analysis, we construct the unique non-equilibrium steady state (NESS). This state depends on β_{L} and β_{R}, but not on the choice of the decoupling parameter M. We prove that in the non-equilibrium case, $\beta_{\mathrm{L}} \neq \beta_{\mathrm{R}}$, this state has *strictly positive* entropy production.

KEY WORDS: XY chain; Jordan–Wigner transformation; non-equilibrium steady state; Bogoliubov automorphism; scattering theory.

1. INTRODUCTION

"Exactly solvable" models—allowing some thermodynamic potential (e.g., pressure, Helmholtz free energy, or ground state energy) to be explicitly computed—have been an essential tool in the development of equilibrium statistical mechanics, especially for our understanding of critical phenomena. Out of equilibrium, the situation is complicated by the fact that the *dynamics* starts to play a central role: Very few non-trivial, exactly solvable models are "integrable" in the sense that their dynamics can also be described in a sufficiently explicit way. The one-dimensional XY model is one of the simplest examples of such an integrable system. It describes a chain of quantum spins with anisotropic nearest neighbor coupling. The key to its "exact solution" is the *Jordan–Wigner transformation* which maps this spin model onto a one-dimensional free Fermi gas (see refs. 18 and 19).

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The implementation of the Jordan–Wigner transformation in the two-sided XY chain model in the framework of C^*-dynamical systems is due to Araki.[1] The integrability of the model might be traced back to the existence of an infinite family of commuting first integrals (so called "master symmetries") first discovered by Barouch and Fuchssteiner.[7] Master symmetries of the XY model have been studied in the C^*-algebraic framework by Araki[2] and Matsui.[20]

This question of integrability appears to be related to transport properties: For finite systems, the overlap of the current with conserved charges prevents the current-current correlations to decay to zero. This ensures the finiteness of the Drude weight which leads, in turn, to an *ideal* thermal conductivity.[11, 30, 31] [2] Numerical investigations also support this fact: A spin chain in contact with heat reservoirs at different temperatures was found to violate the *Fourier law* of diffusive heat conduction as soon as the system was made integrable by a suitable choice of a critical parameter.[26] Moreover, the energy transport via spin $1/2$ excitations was *experimentally* found to contradict the diffusive scenario in the chain direction in Heisenberg-like systems using different materials such as, e.g., $SrCuO_2$ or Sr_2CuO_3 which are often considered as the best physical realizations of the spin $1/2$ Heisenberg chain, see refs. 28 and 29. These highly unusual transport properties in low-dimensional magnetic systems strongly suggest the study of their non-equilibrium features to which we intend to contribute to some extent in this paper. Furthermore, the XY chain represents probably the simplest, non-trivial testing ground for some general ideas on the mathematical structure of non-equilibrium quantum statistical mechanics. We refer the reader to ref. 16 for a recent review on this subject.

In this paper, we adopt Araki's description of C^*-dynamical systems corresponding to the infinite volume XY chain. Following the strategy proposed by Ruelle in ref. 25, we use scattering theory to construct a family of translation invariant non-equilibrium steady states (NESS). This approach allows us to generalize the results of Araki and Ho[6] to anisotropic chains with external magnetic field. The scattering technique being particularly well adapted to the problem simplifies the construction of NESS. A similar setting has been considered by Dirren and Fröhlich.[13] The simple mathematical structure of the NESS allows us to compute various quantities of physical interest. In this paper, we concentrate on entropy production, as defined in refs. 15 and 17 and show that it is *strictly positive*. In particular, this implies that the NESS carries a non-vanishing

[2] To our knowledge, what remains from this relation in the limit of infinitely extended systems is unknown.

energy current. In a forthcoming paper, we will also discuss correlation functions and thermodynamic properties of these states.

The paper is organized as follows. In Section 2, we briefly introduce the model and formulate our main results. Section 3 introduces Araki's description of the underlying C^*-dynamical system in terms of Bogoliubov automorphisms on a self-dual CAR algebra. The scattering theory of this dynamical system is developed in Section 4. Finally, Section 5 contains the proofs of our main results.

2. MODEL AND RESULTS

In order to formulate our results, we start this section with a brief informal description of the XY chain. We refer to Section 3 for a more precise discussion. To each site x of the lattice $\mathbb{Z}$ we attach a copy $\mathscr{H}_{\{x\}}$ of the Hilbert space $\mathbb{C}^2$. We denote by $\sigma_1^{(x)}$, $\sigma_2^{(x)}$, and $\sigma_3^{(x)}$ the Pauli matrices acting on $\mathscr{H}_{\{x\}}$. Polynomials in these Pauli matrices are called local observables. They generate a C^*-algebra which we denote by $\mathfrak{S}$. A state of the system is a linear functional ω on $\mathfrak{S}$ such that $\omega(A^*A) \geqslant 0$ for all $A \in \mathfrak{S}$ and $\omega(\mathbf{1}) = 1$.

For $\gamma \in]-1, 1[$ and $\lambda \in \mathbb{R}$, the formal expression

$$H = -\tfrac{1}{4} \sum_{x \in \mathbb{Z}} \left((1+\gamma)\, \sigma_1^{(x)} \sigma_1^{(x+1)} + (1-\gamma)\, \sigma_2^{(x)} \sigma_2^{(x+1)} + 2\lambda \sigma_3^{(x)}\right) \tag{2.1}$$

has well defined commutators with local observables. The operator

$$A \mapsto \delta(A) \equiv i[H, A],$$

generates a dynamics on $\mathfrak{S}$ which is formally given by $\tau^t(A) = \mathrm{e}^{itH} A\, \mathrm{e}^{-itH}$. More precisely, for any local observable A, one has

$$\frac{d}{dt} \tau^t(A) = \tau^t(\delta(A)). \tag{2.2}$$

The dynamical system $(\mathfrak{S}, \tau)$ describes an infinite chain of spins in which each single spin interacts with its two nearest neighbors and with an external magnetic field λ. The parameter γ controls the anisotropy of the spin-spin coupling (see Fig. 1).

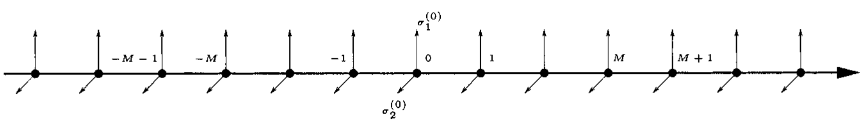

Fig. 1. The *XY* chain.

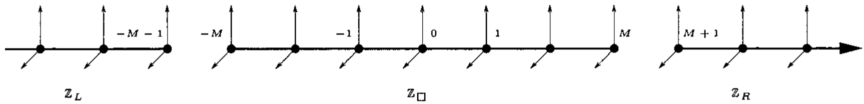

Fig. 2. The decoupled XY chain.

Removing the spin-spin coupling across the bonds $(x, x+1) = (-M-1, -M)$ and $(M, M+1)$ from the sum (2.1) defines a new Hamiltonian H_0, and hence a new dynamics τ_0. According to the decomposition of the lattice into three disjoint pieces $\mathbb{Z} = \mathbb{Z}_{\mathrm{L}} \cup \mathbb{Z}_{\square} \cup \mathbb{Z}_{\mathrm{R}}$ (see Fig. 2), H_0 is the sum of three mutually commuting terms, $H_0 = H_{\mathrm{L}} + H_{\square} + H_{\mathrm{R}}$. The dynamical system $(\mathfrak{S}, \tau_0)$ factorizes into three noninteracting subsystems. We shall denote by $(\mathfrak{S}_{\mathrm{L}}, \tau_{\mathrm{L}})$,... the corresponding subsystems. Note that since $V \equiv H - H_0$ is a local observable, the dynamics τ is a nice perturbation of the decoupled one τ_0.

For each $\beta \in \mathbb{R}$, the dynamical system $(\mathfrak{S}_{\mathrm{L}}, \tau_{\mathrm{L}})$ has a unique equilibrium state $\omega_{\mathrm{L}}^{\beta}$ at temperature β^{-1} (in technical terms, this state is $(\tau_{\mathrm{L}}, \beta)$-KMS). We define $\omega_{\mathrm{R}}^{\beta}$ in a similar way and denote by $\omega_{\square}$ the normalized trace on the (finite dimensional) algebra $\mathfrak{S}_{\square}$. We consider the family of τ_0-invariant states on $\mathfrak{S}$ defined by

$$\omega_0^{M, \beta_{\mathrm{L}}, \beta_{\mathrm{R}}} \equiv \omega_{\mathrm{L}}^{\beta_{\mathrm{L}}} \otimes \omega_{\square} \otimes \omega_{\mathrm{R}}^{\beta_{\mathrm{R}}} \tag{2.3}$$

as initial data for the dynamical system $(\mathfrak{S}, \tau)$. Thus, the two infinite half-chains play the role of thermal reservoirs to which a finite subsystem is attached via the coupling V.

Following Ruelle[24] (see also ref. 16), we say that a state μ is a non-equilibrium steady state (NESS) of τ associated to the initial state ω if, for a sequence $T_n \to +\infty$,

$$\mu(A) = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \omega(\tau^t(A))\, dt,$$

for all $A \in \mathfrak{S}$. We denote by $\Sigma_+(\omega)$ the set of such states. It is easy to check that the elements of this set are indeed τ-invariant states. Moreover, due to the fact that the set of states on $\mathfrak{S}$ is weak-$*$ compact, the set $\Sigma_+(\omega)$ is not empty.

Our first result shows that a unique NESS is associated to initial states of the form (2.3).

Theorem 2.1. For any real β_{L}, β_{R} and for any finite M, the C^*-dynamical system $(\mathfrak{S}, \tau)$ has a unique NESS corresponding to the initial state (2.3),

$$\Sigma_+(\omega_0^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}) = \{\omega_+^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}\}.$$

Moreover, this state is attracting in the sense that

$$\lim_{t\to+\infty} \omega_0^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}(\tau^t(A)) = \omega_+^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}(A), \tag{2.4}$$

for any $A \in \mathfrak{S}$.

Remark 1. In the special case $\beta_{\mathrm{L}} = \beta_{\mathrm{R}}$, the previous Theorem has been proved by Araki in ref. 1. Then, the state $\omega_+^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}$ is the unique (τ, β)-KMS state, and Eq. (2.4) expresses the property of return to equilibrium of the XY chain. For $\beta_{\mathrm{L}} \neq \beta_{\mathrm{R}}$, $\gamma = 0$, and $\lambda = 0$, the state $\omega_+^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}$ has been constructed by Araki and Ho in ref. 6.

To formulate our next result, let us introduce the (2×2)-matrix valued, 2π-periodic function

$$T_+(\xi) \equiv \frac{1}{1+\mathrm{e}^{-(\beta h(\xi)+\delta k(\xi))}},$$

where

$$h(\xi) \equiv (\cos\xi - \lambda)\,\sigma_3 - \gamma\sin\xi\,\sigma_2,$$

and

$$k(\xi) \equiv \mathrm{sign}(\kappa(\xi))\,|h(\xi)|, \tag{2.5}$$

with

$$\kappa(\xi) \equiv 2\lambda\sin\xi - (1-\gamma^2)\sin 2\xi,$$

$$|h(\xi)| = \sqrt{(\cos\xi-\lambda)^2 + \gamma^2\sin^2\xi}.$$

The parameters β and δ are related to the initial temperatures of the chain by the relations

$$\beta \equiv \frac{\beta_{\mathrm{R}} + \beta_{\mathrm{L}}}{2},$$

$$\delta \equiv \frac{\beta_{\mathrm{R}} - \beta_{\mathrm{L}}}{2}.$$

For $x, y \in \mathbb{Z}$ such that $x < y$ and $\mu, \nu \in \{1, 2\}$, we also define the local observables

$$S_{\mu\nu}(x, y) \equiv \sigma_\mu^{(x)} \sigma_3^{(x+1)} \cdots \sigma_3^{(y-1)} \sigma_\nu^{(y)} \in \mathfrak{S},$$

and the set of (2×2)-matrices

$$s_{11} \equiv -(\sigma_3 + i\sigma_2), \qquad s_{12} \equiv \sigma_2(\sigma_3 - i\sigma_2),$$

$$s_{21} \equiv \sigma_2(\sigma_3 + i\sigma_2), \qquad s_{22} \equiv -(\sigma_3 - i\sigma_2).$$

Finally, we recall a few basic definitions. Let ω be a state on $\mathfrak{S}$ and $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ the corresponding GNS-representation of $\mathfrak{S}$. A state η on $\mathfrak{S}$ is called ω-normal if there is a density matrix ρ on $\mathscr{H}_\omega$ such that $\eta(\cdot) = \mathrm{Tr}(\rho\pi_\omega(\cdot))$. η is called ω-singular if $\eta \geqslant \lambda\phi$ for some $\lambda \geqslant 0$ and some ω-normal state ϕ implies $\lambda = 0$. Any ω-normal state η has a unique normal extension to the enveloping von Neumann algebra $\mathfrak{M}_\omega = \pi_\omega(\mathfrak{S})''$. The state ω is called modular if this extension is faithful, i.e., if $A\Omega_\omega = 0$ implies $A = 0$ for any $A \in \mathfrak{M}_\omega$. ω is called primary if $\mathfrak{M}_\omega$ is a factor, i.e., if its center $\mathfrak{M}_\omega \cap \mathfrak{M}'_\omega$ consists of multiples of the identity.

Theorem 2.2. The NESS $\omega_+^{M, \beta_\mathrm{L}, \beta_\mathrm{R}}$ is independent of M, we therefore drop the reference to M in our notation. The state $\omega_+^{\beta_\mathrm{L}, \beta_\mathrm{R}}$ is translation invariant, primary and modular. It is a KMS state for τ if and only if $\beta_\mathrm{L} = \beta_\mathrm{R}$. In a sense to be made precise in Section 3, this state is characterized by the correlation functions

$$\omega_+^{\beta_\mathrm{L}, \beta_\mathrm{R}}(\sigma_3^{(x)}) = -\int_0^{2\pi} \frac{d\xi}{2\pi} \, \mathrm{tr}[\sigma_3 T_+(\xi)],$$

$$\omega_+^{\beta_\mathrm{L}, \beta_\mathrm{R}}(S_{\mu\nu}(x, y)) = -\int_0^{2\pi} \frac{d\xi}{2\pi} \, \mathrm{tr}[s_{\mu\nu} T_+(\xi)] \, \mathrm{e}^{-i(x-y)\,\xi}.$$

Remark 2. In technical terms, the NESS $\omega_+^{\beta_\mathrm{L}, \beta_\mathrm{R}}$ is a translation invariant, quasi-free state on the CAR algebra obtained from $\mathfrak{S}$ by a Jordan–Wigner transformation. See Section 3 for details. The proof of Theorem 2.2 is given in Section 5, and follows Ruelle's scattering approach (see refs. 25 and also 16).

Remark 3. In a more informal way, the NESS $\omega_+^{\beta_\mathrm{L}, \beta_\mathrm{R}}$ can be described as an equilibrium state, at temperature β^{-1}, for the Hamiltonian $H_{\mathrm{NESS}} \equiv H + \frac{\delta}{\beta} K$, with a *long range,* multi-body interaction

$$K \equiv \frac{1}{2i} \sum_{x<y} \check{k}(x-y)(S_{21}(x, y) - S_{12}(x, y)),$$

where $\check{k}(x)$ denotes the inverse Fourier transform of $k(\xi)$. Note that, due to the singularity of the sign function in Eq. (2.5), $\check{k}(x)$ is long range. See also the remark following Corollary 4.4.

Entropy production in quantum spin systems has been discussed recently by Ruelle in refs. 24 and 25. A more general approach, based on the rate of increase of the relative entropy of the state $\omega_0 \circ \tau^t$ with respect to the initial state ω_0, has been proposed in ref. 15 (see also ref. 17). For the XY chain, these two definitions coincide: The entropy production in a NESS $\omega \in \Sigma_+(\omega_0^{M,\beta_{\rm L},\beta_{\rm R}})$ is given by

$$\mathrm{Ep}(\omega) \equiv \beta_{\rm L}\, \omega(\Phi_{\rm L}) + \beta_{\rm R}\, \omega(\Phi_{\rm R}),$$

where $\Phi_{\rm L}$ and $\Phi_{\rm R}$ are the heat fluxes leaving the infinite reservoirs $\mathbb{Z}_{\rm L}$ and $\mathbb{Z}_{\rm R}$ and entering the finite box $\mathbb{Z}_\square$. The formal expression for these fluxes is

$$\Phi_{\rm L} = -i[H, H_{\rm L}], \qquad \Phi_{\rm R} = -i[H, H_{\rm R}].$$

Due to the local structure of the Hamiltonians, these expressions make sense, even though the Hamiltonians themselves are ill-defined. For example the flux from the right reservoir is easily seen to be given by

$$\begin{aligned}\Phi_{\rm R} &= i[H_{\rm R}, V]\\ &= -\frac{i}{4}\left[(1+\gamma)\,\sigma_1^{(M+1)}\sigma_1^{(M+2)} + (1-\gamma)\,\sigma_2^{(M+1)}\sigma_2^{(M+2)} + 2\lambda\sigma_3^{(M+1)}, V\right],\end{aligned}$$

a local observable which can be further expressed as

$$\begin{aligned}\Phi_{\rm R} = {}& \frac{1-\gamma^2}{8}\,(S_{12}(M, M+2) - S_{21}(M, M+2))\\ & -\lambda\left(\frac{1+\gamma}{4}\,S_{12}(M, M+1) - \frac{1-\gamma}{4}\,S_{21}(M, M+1)\right).\end{aligned} \tag{2.6}$$

Moreover, since $\Phi_{\rm L} + \Phi_{\rm R} = \delta(H_\square + V)$ and $H_\square + V$ is a local observable, it follows from Eq. (2.2) that

$$\omega(\Phi_{\rm L}) + \omega(\Phi_{\rm R}) = 0, \tag{2.7}$$

for any stationary state ω. This is an expression of the first law of thermodynamics (energy conservation). Note however that $\omega(\Phi_{\rm R})$ does not necessarily vanish since we can not apply (2.2) to the formal observable $H_{\rm R}$. The second law of thermodynamics can be written as

$$\mathrm{Ep}(\omega) \geqslant 0. \tag{2.8}$$

This inequality has been proved in refs. 15 and 25 for any NESS $\omega \in \Sigma_+(\omega_0)$. In particular we can rewrite the entropy production as

$$\mathrm{Ep}(\omega) = 2\delta\, \omega(\Phi_{\mathrm{R}}), \tag{2.9}$$

and Inequality (2.8) states that energy is flowing through the finite box $\mathbb{Z}_\square$, from the hotter half-chain to the colder one.

It follows from the next result that the entropy production in the state $\omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}$ is strictly positive, provided $\beta_{\mathrm{L}} \neq \beta_{\mathrm{R}}$. In particular, this state carries a non-vanishing energy current.

Theorem 2.3. The entropy production in the NESS $\omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}$ is given by

$$\mathrm{Ep}(\omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}) = \frac{\delta}{4}\int_0^{2\pi}\frac{d\xi}{2\pi}\,|\kappa|\,\frac{\mathrm{sh}\,\delta\,|h|}{\mathrm{ch}^2(\beta\,|h|/2)+\mathrm{sh}^2(\delta\,|h|/2)}.$$

Remark 4. One easily checks that the heat current $\omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}(\Phi_{\mathrm{R}})$ is a strictly monotonic function of the temperature difference $T_{\mathrm{R}} - T_{\mathrm{L}}$.

The positivity of entropy production has the following consequence on the mathematical nature of the NESS.

Corollary 2.4. If $\beta_{\mathrm{L}} \neq \beta_{\mathrm{R}}$, then the NESS $\omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}$ is $\omega_0^{M,\beta_{\mathrm{L}},\beta_{\mathrm{R}}}$-singular.

Our last result is about the decay of longitudinal correlations

$$C_3^T(x) \equiv \omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}(\sigma_3^{(0)}\sigma_3^{(x)}) - \omega_+^{\beta_{\mathrm{L}},\beta_{\mathrm{R}}}(\sigma_3^{(0)})^2.$$

At thermal equilibrium ($\beta_{\mathrm{L}} = \beta_{\mathrm{R}}$), the truncated two-point function $C_3^T(x)$ is known to decay exponentially as $x \to \infty$ (see refs. 8 and 21). Since the NESS Hamiltonian H_{NESS} is long range (see Remark 3 in Section 2), we expect these correlations to have a slower decay out of equilibrium.

Theorem 2.5. The truncated longitudinal two-point function $C_3^T(x)$ decays like $|x|^{-2}$ at infinity, i.e.,

$$0 < \limsup_{x^2\to\infty} |x^2\, C_3^T(x)| < \infty.$$

Remark 5. Note that $C_3^T(x)$ is still summable. Therefore, we expect the fluctuations of the magnetization

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{x \in \Lambda} (\sigma_3^{(x)} - \omega_+^{\beta_{\mathrm{L}}, \beta_{\mathrm{R}}}(\sigma_3^{(0)})),$$

to satisfy a central limit theorem as $\Lambda \uparrow \mathbb{Z}$.

3. THE *XY* DYNAMICS OF THE INFINITE SPIN CHAIN

In this section, we briefly summarize a few well known facts about the one-dimensional XY chain and the corresponding C^*-dynamical system. Following Araki,[1] we describe this dynamical system as a group of Bogoliubov automorphisms on a self-dual CAR algebra.

3.1. Kinematics

To each finite subset Λ of the lattice $\mathbb{Z}$ we associate the Hilbert space

$$\mathscr{H}_\Lambda \equiv \bigotimes_{x \in \Lambda} \mathscr{H}_{\{x\}},$$

where $\mathscr{H}_{\{x\}} \equiv \mathbb{C}^2$. The corresponding set of observables is the full matrix algebra

$$\mathfrak{S}_\Lambda \equiv \mathscr{B}(\mathscr{H}_\Lambda).$$

For an arbitrary subset $\mathscr{L} \subset \mathbb{Z}$, the union

$$\mathfrak{S}_{\mathscr{L}}^0 \equiv \bigcup_{\Lambda \subset \mathscr{L}} \mathfrak{S}_\Lambda,$$

over all finite subsets of $\mathscr{L}$ is a $*$-algebra equipped with a C^*-norm. Its completion

$$\mathfrak{S}_{\mathscr{L}} \equiv \overline{\mathfrak{S}_{\mathscr{L}}^0}, \tag{3.10}$$

is a C^*-algebra: The infinite tensor product of the $\mathscr{B}(\mathscr{H}_{\{x\}})$ for $x \in \mathscr{L}$. Moreover, if $\mathscr{L}$ and $\mathscr{L}'$ are disjoint subsets of $\mathbb{Z}$, then the C^*-tensor product of $\mathfrak{S}_{\mathscr{L}}$ and $\mathfrak{S}_{\mathscr{L}'}$ is

$$\mathfrak{S}_{\mathscr{L}} \otimes \mathfrak{S}_{\mathscr{L}'} = \mathfrak{S}_{\mathscr{L} \cup \mathscr{L}'},$$

(see Section 2.6 in ref. 9 and Section I.23 in ref. 27 for details). Since the Pauli matrices

$$\sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

generate $\mathscr{B}(\mathscr{H}_{\{x\}})$, $\mathfrak{S}_\Lambda$ is the algebra of polynomials in the matrices

$$\sigma_\alpha^{(x)} \equiv \cdots \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_\alpha \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots,$$

where the factor σ_α acts on $\mathscr{H}_{\{x\}}$, and $x \in \Lambda$. As a consequence of (3.10), any element of $\mathfrak{S}_{\mathscr{L}}$ is a uniform limit of polynomials in the matrices $\sigma_\alpha^{(x)}$, with $x \in \mathscr{L}$. In particular, $\mathfrak{S}_{\mathbb{Z}}$ describes the kinematical structure of an infinite chain of spins. In the following, we call $\mathfrak{S}_{\mathbb{Z}}$ the spin algebra, and denote it by $\mathfrak{S}$. We also fix $M \geqslant 0$ and set

$$\mathfrak{S}_{\mathrm{L}} \equiv \mathfrak{S}_{\{x < -M\}}, \qquad \mathfrak{S}_{\square} \equiv \mathfrak{S}_{\{-M \leqslant x \leqslant M\}}, \qquad \mathfrak{S}_{\mathrm{R}} \equiv \mathfrak{S}_{\{x > M\}}.$$

3.2. Dynamics

For any finite $\Lambda \subset \mathbb{Z}$, the local XY Hamiltonian is defined by

$$H_\Lambda \equiv \sum_{X \subset \Lambda} \phi(X), \tag{3.11}$$

where the interaction ϕ is given by

$$\phi(X) = \begin{cases} -\frac{1}{2}\,\lambda\sigma_3^{(x)}, & X = \{x\}, \\ -\frac{1}{4}\,\{(1+\gamma)\,\sigma_1^{(x)}\sigma_1^{(x+1)} + (1-\gamma)\,\sigma_2^{(x)}\sigma_2^{(x+1)}\}, & X = \{x, x+1\}, \\ 0, & \text{otherwise.} \end{cases}$$

The parameters λ (magnetic field strength) and γ (anisotropy) satisfy

$$\lambda \in \mathbb{R}, \qquad \gamma \in\,]-1, 1[.$$

Clearly, H_Λ is a self-adjoint element of $\mathfrak{S}_\Lambda$, and the formula

$$\tau_\Lambda^t(A) \equiv \mathrm{e}^{itH_\Lambda} A\, \mathrm{e}^{-itH_\Lambda},$$

defines a norm continuous group of $*$-automorphisms of $\mathfrak{S}$. Since the interaction ϕ has finite range, the thermodynamic limit

$$\tau^t(A) \equiv \lim_{\Lambda \uparrow \mathbb{Z}} \tau_\Lambda^t(A)$$

exists in norm for all $A \in \mathfrak{S}$. Therefore, τ^t is a strongly continuous group of $*$-automorphisms of $\mathfrak{S}$ (see Theorem 6.2.4 in ref. 10). The C^*-dynamical system describing the infinite XY chain is $(\mathfrak{S}, \tau)$.

We denote by δ the generator of the time evolution τ. For any finite $\Lambda \subset \mathbb{Z}$ we have $\mathfrak{S}_\Lambda \subset D(\delta)$, and for $A \in \mathfrak{S}_\Lambda$,

$$\delta(A) = i[H_{\Lambda'}, A],$$

provided $\Lambda' \supset \{x \in \mathbb{Z} \mid \mathrm{dist}(x, \Lambda) \leqslant 1\}$.

According to standard time dependent perturbation theory, any self-adjoint $P \in \mathfrak{S}$ induces a perturbed time evolution generated by $\delta(\cdot) = \delta_0(\cdot) + i[P, \cdot]$, with $D(\delta) = D(\delta_0)$. The perturbed dynamics is given by the norm convergent Dyson expansion

$$\tau^t(A) = \mathrm{e}^{t\delta}(A) = \tau_0^t(A) + \sum_{n \geqslant 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, [\tau_0^{t_n}(P), [\ldots, [\tau_0^{t_1}(P), \tau_0^t(A)] \cdots]],$$

and defines a perturbed C^*-dynamical system $(\mathfrak{S}, \tau)$ (see ref. 10, Section 5.4 for details). Of particular interest to us is the perturbation

$$V \equiv \phi(\{-M-1, -M\}) + \phi(\{M, M+1\}),$$

which decouples the full XY dynamics τ^t as

$$\tau_0^t = \tau_{\mathrm{L}}^t \otimes \tau_{\square}^t \otimes \tau_{\mathrm{R}}^t,$$

according to the factorization

$$\mathfrak{S} = \mathfrak{S}_{\mathrm{L}} \otimes \mathfrak{S}_{\square} \otimes \mathfrak{S}_{\mathrm{R}}.$$

Let θ be the $*$-automorphism of $\mathfrak{S}$ which rotates all spins around the z-axis by an angle of π,

$$\theta(\sigma_1^{(x)}) = -\sigma_1^{(x)}, \qquad \theta(\sigma_2^{(x)}) = -\sigma_2^{(x)}, \qquad \theta(\sigma_3^{(x)}) = \sigma_3^{(x)}. \tag{3.12}$$

Since θ is an involution, it induces a decomposition $\mathfrak{S} = \mathfrak{S}_+ + \mathfrak{S}_-$ into even and odd subspaces,

$$\mathfrak{S}_\pm \equiv \{A \in \mathfrak{S} \mid \theta(A) = \pm A\}.$$

Note that $\mathfrak{S}_+$ is a C^*-subalgebra of $\mathfrak{S}$. Since $\theta(\phi(X)) = \phi(X)$, $\mathfrak{S}_\pm$ are invariant under τ_0^t and τ^t.

For $\beta \in \mathbb{R}$, let $\omega_{\mathrm{L}}^{\beta}$ be the unique $(\tau_{\mathrm{L}}, \beta)$-KMS state on $\mathfrak{S}_{\mathrm{L}}$. We define similarly $\omega_{\mathrm{R}}^{\beta}$, and denote by $\omega_{\square}$ the normalized trace on $\mathfrak{S}_{\square}$ (i.e., the unique $(\tau_{\square}, 0)$-KMS state). By the uniqueness of these states and the fact that θ commutes with the decoupled dynamics τ_0^t, the state

$$\omega_0^{M, \beta_{\mathrm{L}}, \beta_{\mathrm{R}}} = \omega_{\mathrm{L}}^{\beta_{\mathrm{L}}} \otimes \omega_{\square} \otimes \omega_{\mathrm{R}}^{\beta_{\mathrm{R}}}, \tag{3.13}$$

vanishes on the odd subspace $\mathfrak{S}_-$. Therefore, to study the limit

$$\omega_+^{\beta_{\mathrm{L}}, \beta_{\mathrm{R}}}(A) \equiv \lim_{t \to +\infty} \omega_0^{M, \beta_{\mathrm{L}}, \beta_{\mathrm{R}}} \circ \tau^t(A),$$

it suffices to consider $A \in \mathfrak{S}_+$. In other words, only the even part $(\mathfrak{S}_+, \tau)$ of the XY dynamical system is of interest to us.

3.3. The Fermionic Picture

For computational purposes, we shall now follow[1] and apply a Jordan–Wigner transformation to map the spin algebra $\mathfrak{S}_+$ into a more convenient CAR algebra. Let $\mathfrak{A}$ be the C^*-algebra generated by $\mathfrak{S}$ and an element $T \notin \mathfrak{S}$ satisfying

$$T = T^*, \qquad T^2 = \mathbf{1}, \qquad TA = \theta_-(A)\, T, \tag{3.14}$$

where θ_- is the $*$-automorphism of $\mathfrak{S}$ given by

$$\theta_-(\sigma_\alpha^{(x)}) \equiv \begin{cases} -\sigma_\alpha^{(x)}, & \alpha = 1, 2, \quad x \leqslant 0, \\ \sigma_\alpha^{(x)}, & \alpha = 1, 2, \quad x > 0, \\ \sigma_\alpha^{(x)}, & \alpha = 3, \quad x \in \mathbb{Z}. \end{cases}$$

It follows immediately from (3.14) that the enlarged algebra $\mathfrak{A}$ can be written as

$$\mathfrak{A} = \mathfrak{S} + T\mathfrak{S}.$$

For any $x \in \mathbb{Z}$, define $S^{(x)} \in \mathfrak{S}_+$ by the formula

$$S^{(x)} \equiv \begin{cases} \sigma_3^{(1)} \cdots \sigma_3^{(x-1)}, & x > 1, \\ \mathbf{1}, & x = 1, \\ \sigma_3^{(x)} \cdots \sigma_3^{(0)}, & x < 1. \end{cases}$$

A simple calculation shows that the elements of $T\mathfrak{S}_-$ defined by

$$a_x \equiv TS^{(x)}(\sigma_1^{(x)} - i\sigma_2^{(x)})/2, \qquad a_x^* = TS^{(x)}(\sigma_1^{(x)} + i\sigma_2^{(x)})/2, \tag{3.15}$$

are fermionic annihilation and creation operators: They satisfy the canonical anticommutation relations (CAR)

$$\{a_x, a_y\} = 0, \qquad \{a_x^*, a_y^*\} = 0, \qquad \{a_x, a_y^*\} = \delta_{x,y}, \tag{3.16}$$

where $\{A, B\} \equiv AB + BA$. We denote by $\mathfrak{F}$ the C^*-subalgebra of $\mathfrak{A}$ generated by these annihilation and creation operators, and we remark that

$$\mathfrak{F} \subset \mathfrak{S}_+ + T\mathfrak{S}_-. \tag{3.17}$$

Extending the $*$-automorphism θ to $\mathfrak{A}$ by setting

$$\theta(T) \equiv T,$$

yields the decomposition $\mathfrak{F} = \mathfrak{F}_+ + \mathfrak{F}_-$. Moreover, from Eq. (3.15), we obtain that a_x and a_x^* are odd. Note that the relations (3.15) are easily inverted to give

$$\sigma_1^{(x)} = TS^{(x)}(a_x + a_x^*), \quad \sigma_2^{(x)} = i\,TS^{(x)}(a_x - a_x^*), \quad \sigma_3^{(x)} = 2a_x^* a_x - 1, \tag{3.18}$$

from which we conclude that

$$\mathfrak{S} \subset \mathfrak{F}_+ + T\mathfrak{F}_-. \tag{3.19}$$

The two inclusions (3.17) and (3.19) finally yield

$$\mathfrak{S}_+ = \mathfrak{F}_+, \qquad \mathfrak{S}_- = T\mathfrak{F}_-.$$

In particular, we have $\phi(X) \in \mathfrak{F}_+$, and a simple calculation leads to the following explicit formulae

$$\phi(X) = \begin{cases} -\frac{1}{2}\lambda(2a_x^* a_x - 1), & X = \{x\}, \\ \frac{1}{2}\{a_x^* a_{x+1} + a_{x+1}^* a_x + \gamma(a_x^* a_{x+1}^* + a_{x+1} a_x)\}, & X = \{x, x+1\}, \\ 0, & \text{otherwise}. \end{cases}$$

3.4. The Bogoliubov Automorphism

Let $\mathfrak{h} \equiv \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \simeq \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ and define the linear map

$$\mathfrak{h} \ni f \equiv (f_+, f_-) \mapsto B(f) \equiv \sum_{x \in \mathbb{Z}} (f_+(x)\, a_x^* + f_-(x)\, a_x) \in \mathfrak{F}_-.$$

It follows from the CAR (3.16) that this sum converges in the C^*-norm of $\mathfrak{F}$, and that

$$\{B^*(f), B(g)\} = (f, g)\,\mathbf{1}, \tag{3.20}$$

where we have set $B^*(f) \equiv B(f)^*$. Moreover,

$$B^*(f) = B(Jf),$$

where J is the antiunitary involution on $\mathfrak{h}$ defined by

$$J{:}(f_+, f_-) \mapsto (\bar{f}_-, \bar{f}_+).$$

Clearly, $\mathfrak{F}$ is the C^*-algebra generated by polynomials in $B(f)$, and, since $\theta(B(f)) = -B(f)$, the even part $\mathfrak{F}_+$ is generated by even polynomials in the $B(f)$. Thus $\mathfrak{F}$ is a self-dual CAR algebra as introduced by Araki in ref. 3 (see also refs. 5 and 14).

To a finite rank operator $k \equiv \sum_{j=1}^{n} f_j\,(g_j, \cdot)$ on $\mathfrak{h}$, $f_j, g_j \in \mathfrak{h}$, we associate

$$\mathbf{B}(k) \equiv \sum_{j=1}^{n} B(f_j)\, B^*(g_j) \in \mathfrak{F}_+, \tag{3.21}$$

which is easily seen to depend only on k and not on its representation in terms of g_j and f_j. The following properties are immediate consequences of this definition and of the CAR (3.20):

$$\begin{aligned}
\mathbf{B}(k^*) &= \mathbf{B}(k)^*,\\
\mathbf{B}(k + j(k)) &= \mathrm{tr}(k)\,\mathbf{1},\\
[\mathbf{B}(k), B(f)] &= B((k - j(k))\, f),\\
[\mathbf{B}(k), \mathbf{B}(k')] &= \mathbf{B}([k - j(k), k' - j(k')])/2,
\end{aligned}$$

where $j(k) \equiv Jk^*J$. In particular, if $k + j(k) = 0$, one has

$$\mathrm{e}^{t\mathbf{B}(k)/2}\, B(f)\, \mathrm{e}^{-t\mathbf{B}(k)/2} = B(\mathrm{e}^{tk} f). \tag{3.22}$$

The local Hamiltonian H_Λ can be expressed as

$$H_\Lambda = \mathbf{B}(h_\Lambda)/2, \qquad h_\Lambda = \sum_{X \subset \Lambda} \varphi(X),$$

where the non-vanishing φ are given by

$$\varphi(\{x\}) \equiv -\lambda\, |x\rangle\langle x| \otimes \sigma_3, \qquad \varphi(\{x, x+1\}) \equiv c_x \otimes \sigma_3 - \gamma s_x \otimes \sigma_2,$$

with

$$c_x \equiv (|x\rangle\langle x+1| + |x+1\rangle\langle x|)/2, \qquad s_x \equiv (|x\rangle\langle x+1| - |x+1\rangle\langle x|)/2i.$$

Since $\varphi(X) + j(\varphi(X)) = 0$, the local dynamics τ_Λ extends from $\mathfrak{S}_+$ to a Bogoliubov automorphism of the self-dual CAR algebra $\mathfrak{F}$

$$\tau_\Lambda^t(B(f)) = B(\mathrm{e}^{ith_\Lambda} f).$$

The limit $\Lambda \uparrow \mathbb{Z}$ leads to

$$\tau^t(B(f)) = B(\mathrm{e}^{ith} f),$$

where the translation invariant Hamiltonian $h \equiv \sum_{X \subset \mathbb{Z}} \varphi(X)$ is given, in the Fourier representation $\mathfrak{h} = L^2(S^1, \frac{d\xi}{2\pi}) \otimes \mathbb{C}^2$, by the formula

$$h \equiv (\cos\xi - \lambda) \otimes \sigma_3 - \gamma \sin\xi \otimes \sigma_2. \tag{3.23}$$

The decoupled dynamics τ_0 is implemented in a similar way in the self-dual CAR algebra. The corresponding Hamiltonian h_0 decouples according to the decomposition

$$\mathfrak{h} = \ell^2(\mathbb{Z}_{\mathrm{L}}) \otimes \mathbb{C}^2 \oplus \ell^2(\mathbb{Z}_\square) \otimes \mathbb{C}^2 \oplus \ell^2(\mathbb{Z}_{\mathrm{R}}) \otimes \mathbb{C}^2.$$

It is given by

$$h_0 \equiv h - v = h_{\mathrm{L}} \oplus h_\square \oplus h_{\mathrm{R}}, \tag{3.24}$$

where the finite rank perturbation v looks like

$$v \equiv \varphi(\{-M-1, -M\}) + \varphi(\{M, M+1\}). \tag{3.25}$$

3.5. Quasi-Free States

A quasi-free state on $\mathfrak{F}$ is a state ω which vanishes on $\mathfrak{F}_-$ and satisfies the Wick expansion formula

$$\omega(B(f_1) \cdots B(f_{2n})) = \sum_\pi \mathrm{sign}(\pi) \prod_{k=1}^{n} \omega(B(f_{\pi(2k-1)})\, B(f_{\pi(2k)})), \tag{3.26}$$

where the sum runs over all permutations $\pi \in S_{2n}$, with signature $\mathrm{sign}(\pi)$, such that

$$\pi(2k),\, \pi(2k+1) > \pi(2k-1).$$

Such a state is completely characterized by its two-point function $\omega(B^*(f)\, B(g))$, which in turn determines a bounded operator T on $\mathfrak{h}$ such that

$$(f, Tg) \equiv \omega(B^*(f)\, B(g)).$$

Any self-adjoint operator T on $\mathfrak{h}$ such that

$$0 \leqslant T \leqslant I, \qquad T + j(T) = I, \tag{3.27}$$

determines in this way a unique quasi-free state on $\mathfrak{F}$ (see ref. 3). Let k be a self-adjoint operator on $\mathfrak{h}$ such that $k + j(k) = 0$, and τ_k the corresponding group of Bogoliubov automorphisms of $\mathfrak{F}$ (i.e., $\tau_k^t(B(f)) = B(\mathrm{e}^{itk} f)$). Then, $(1+\mathrm{e}^{\beta k})^{-1}$ satisfies Condition (3.27) and the corresponding quasi-free state is (τ_k, β)-KMS. It follows that the state $\omega_0^{M, \beta_\mathrm{L}, \beta_\mathrm{R}}$ defined in Eq. (3.13), or more precisely its restriction to $\mathfrak{F}_+$, extends to the quasi-free state on $\mathfrak{F}$ determined by

$$T_0 \equiv \frac{1}{1+\mathrm{e}^{k_0}}, \tag{3.28}$$

where

$$k_0 \equiv \beta_\mathrm{L}\, h_\mathrm{L} \oplus 0 \oplus \beta_\mathrm{R}\, h_\mathrm{R}. \tag{3.29}$$

4. SCATTERING THEORY

In this section, we apply Ruelle's scattering approach (see ref. 25) to the construction of non-equilibrium steady states of the C^*-dynamical system $(\mathfrak{F}, \tau)$. Due to the fact that the dynamics is implemented by a group of Bogoliubov automorphisms, the analysis reduces to a simple Hilbert space scattering problem.

Lemma 4.1. For any $A \in \mathfrak{F}$, the norm limit

$$\gamma_+(A) \equiv \lim_{t \to +\infty} \tau_0^{-t} \circ \tau^t(A)$$

exists. The Møller morphism γ_+ is completely characterized by

$$\gamma_+(B(f)) = B(\Omega_- f),$$

for $f \in \mathfrak{h}$, where the wave operator Ω_- is given by

$$\Omega_- \equiv \operatorname{s-}\lim_{t\to+\infty} \mathrm{e}^{-ith_0}\mathrm{e}^{ith}.$$

Proof. It follows from Eq. (3.23) that h has purely absolutely continuous spectrum. Since the perturbation v is finite rank, it follows from Kato–Birman theory that the wave operator Ω_- exists and is complete (i.e., $\operatorname{Ran}\Omega_- = \operatorname{Ran} P_{\mathrm{ac}}(h_0)$, where $P_{\mathrm{ac}}(h_0)$ is the orthogonal projection onto the absolutely continuous spectral subspace of h_0, see for example Theorem XI.8 in ref. 23). From the CAR (3.20), we get the estimate $\|B(f)\| \leqslant \|f\|$, from which we conclude that

$$\tau_0^{-t} \circ \tau^t(B(f)) = B(\mathrm{e}^{-ith_0}\mathrm{e}^{ith} f)$$

converges in norm to $B(\Omega_- f)$. The norm convergence of $\tau_0^{-t} \circ \tau^t(A)$ extends by continuity to all $A \in \mathfrak{F}$. ∎

Since the state $\omega_0^{M,\beta_\mathrm{L},\beta_\mathrm{R}}$ defined in Eq. (3.13) is invariant under the decoupled dynamics τ_0, we have $\omega_0^{M,\beta_\mathrm{L},\beta_\mathrm{R}} \circ \tau^t = \omega_0^{M,\beta_\mathrm{L},\beta_\mathrm{R}} \circ \tau_0^{-t} \circ \tau^t$. Therefore, τ has a unique steady state $\omega_+^{M,\beta_\mathrm{L},\beta_\mathrm{R}}$ corresponding to this initial state,

$$\omega_+^{M,\beta_\mathrm{L},\beta_\mathrm{R}}(A) \equiv \lim_{t\to+\infty} \omega_0^{M,\beta_\mathrm{L},\beta_\mathrm{R}} \circ \tau^t(A) = \omega_0 \circ \gamma_+(A).$$

This clearly proves Theorem 2.1.

The following is an immediate consequence of Lemma 4.1.

Corollary 4.2. The steady state $\omega_+^{M,\beta_\mathrm{L},\beta_\mathrm{R}}$ is quasi-free on $\mathfrak{F}$. Its two-point function is given by

$$\omega_+^{M,\beta_\mathrm{L},\beta_\mathrm{R}}(B^*(f)\,B(g)) = (f, T_+ g) \equiv (f, \Omega_-^* T_0 \Omega_- g). \tag{4.30}$$

We now proceed with the explicit evaluation of the last formula by decomposing the wave operator into left and right components. Let us denote by i_L the natural injection

$$i_\mathrm{L}\colon \ell^2(\mathbb{Z}_\mathrm{L}) \otimes \mathbb{C}^2 \to \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2,$$

and define similarly i_R.

Lemma 4.3. The wave operator Ω_- can be written as

$$\Omega_- = \sum_{\alpha \in \{L, R\}} i_\alpha \, \Omega_\alpha,$$

where the partial wave operators are defined by

$$\Omega_\alpha \equiv \text{s}-\lim_{t \to +\infty} \mathrm{e}^{-ith_\alpha} \, i_\alpha^* \, \mathrm{e}^{ith}.$$

The intertwining property

$$h_\alpha \, \Omega_\alpha = \Omega_\alpha \, h,$$

holds. Moreover, the asymptotic projections

$$P_\alpha \equiv \text{s}-\lim_{t \to +\infty} \mathrm{e}^{-ith} \, i_\alpha i_\alpha^* \, \mathrm{e}^{ith}, \tag{4.31}$$

exist, are given by

$$P_\alpha = \Omega_\alpha^* \Omega_\alpha, \tag{4.32}$$

and satisfy

$$P_\mathrm{L} + P_\mathrm{R} = 1, \qquad [P_\alpha, h] = 0.$$

Proof. Since $h_\alpha i_\alpha^* - i_\alpha^* h$ are finite rank, the existence of the partial wave operators Ω_α and the fact that

$$\Omega_\alpha^* = \text{s}-\lim_{t \to +\infty} \mathrm{e}^{-ith} \, i_\alpha \, \mathrm{e}^{ith_\alpha} P_{\mathrm{ac}}(h_\alpha),$$

follow again from Kato–Birman theory. Eqs. (4.31) and (4.32) are consequences of the chain rule. The fact that the P_α are complementary orthogonal projections commuting with h follows from the Davies–Simon theory.[12] The decomposition of Ω_- follows from the formula

$$i_\alpha \mathrm{e}^{-ith_\alpha} = \mathrm{e}^{-ith_0} i_\alpha,$$

and the fact that $I - i_\mathrm{L} \, i_\mathrm{L}^* - i_\mathrm{R} \, i_\mathrm{R}^*$ is finite rank. ∎

Recall that $\beta \equiv (\beta_\mathrm{R} + \beta_\mathrm{L})/2$ and $\delta \equiv (\beta_\mathrm{R} - \beta_\mathrm{L})/2$.

Corollary 4.4. The operator T_+ defining the steady state $\omega_+^{M, \beta_\mathrm{L}, \beta_\mathrm{R}}$ (see Eq. (4.30)) has the following form

$$T_+ = (1 + \mathrm{e}^{k_+})^{-1}.$$

Moreover,

$$k_+ = \beta h + \delta s h,$$

where $s \equiv P_{\rm R} - P_{\rm L}$.

Remark. Since $k_+ = \beta_{\rm L} h P_{\rm L} \oplus \beta_{\rm R} h P_{\rm R}$, the state $\omega_+^{M,\beta_{\rm L},\beta_{\rm R}}$ describes a mixture of two independent species: Left-movers corresponding to $P_{\rm R}\mathfrak{h}$ are at thermal equilibrium at inverse temperature $\beta_{\rm R}$ and right-movers corresponding to $P_{\rm L}\mathfrak{h}$ are at thermal equilibrium at inverse temperature $\beta_{\rm L}$. The state $\omega_+^{M,\beta_{\rm L},\beta_{\rm R}}$ has a very similar structure to the conducting equilibrium states of quantum wires studied by Alekseev, Cheianov, and Fröhlich in ref. 4. This connection remains to be studied in more details.

Proof. Going back to Eqs. (3.28) and (3.29) and using the fact that $k_0 = \beta_{\rm L} i_{\rm L} h_{\rm L} i_{\rm L}^* + \beta_{\rm R} i_{\rm R} h_{\rm R} i_{\rm R}^*$, we easily obtain from Lemma 4.3 and Eq. (4.30) that

$$T_+ = (1 + \mathrm{e}^{k_+})^{-1},$$

where k_+ is given by the formula

$$k_+ = \beta_{\rm L}\ \Omega_{\rm L}^* h_{\rm L} \Omega_{\rm L} + \beta_{\rm R}\ \Omega_{\rm R}^* h_{\rm R} \Omega_{\rm R} = (\beta_{\rm L}\ P_{\rm L} + \beta_{\rm R}\ P_{\rm R})\, h = \beta h + \delta s h. \quad \blacksquare \tag{4.33}$$

We now relate s to the asymptotic velocity of the dynamics generated by h. Let us introduce the position and velocity operators

$$x \equiv -i\partial_\xi \otimes \mathbf{1}, \qquad p \equiv -i[h, x],$$

on $\mathfrak{h}$. Using a standard notation, we can write

$$h = \underline{h}(\xi) \cdot \underline{\sigma}, \qquad p = \underline{p}(\xi) \cdot \underline{\sigma},$$

where

$$\underline{h}(\xi) \equiv \begin{pmatrix} 0 \\ -\gamma \sin \xi \\ \cos \xi - \lambda \end{pmatrix}, \qquad \underline{p}(\xi) \equiv \begin{pmatrix} 0 \\ -\gamma \cos \xi \\ -\sin \xi \end{pmatrix}.$$

Lemma 4.5. We set $x_t \equiv \mathrm{e}^{-ith} x\, \mathrm{e}^{ith}$. Then, the asymptotic velocity

$$v_- \equiv \lim_{t \to +\infty} \frac{x_t}{t},$$

exists in the strong resolvent sense. Moreover,

$$v_- = \mu h,$$

where $\mu \equiv \underline{p}\cdot\underline{h}/\underline{h}\cdot\underline{h}$. Finally, the following formula holds

$$s \equiv P_{\rm R} - P_{\rm L} = \operatorname{sign} v_-.$$

Proof. By Eq. (4.31), we have

$$s = \operatorname*{s-lim}_{t\to+\infty} \mathrm{e}^{-ith}(i_{\rm R} i_{\rm R}^* - i_{\rm L} i_{\rm L}^*)\, \mathrm{e}^{ith},$$

and since $i_{\rm R} i_{\rm R}^* - i_{\rm L} i_{\rm L}^*$ is equal to $\operatorname{sign}(x)$, up to a finite rank operator, we also have

$$s = \operatorname*{s-lim}_{t\to+\infty} \operatorname{sign}(x_t) = \operatorname*{s-lim}_{t\to+\infty} \operatorname{sign}\left(\frac{x_t}{t}\right).$$

A straightforward calculation gives

$$\dot{x}_t = p_t = \mathrm{e}^{2t\underline{h}\cdot\underline{\Sigma}}\, \underline{p}\cdot\underline{\sigma},$$

where the matrices Σ_α are the generators from the Lie algebra of SO(3), i.e., $(\underline{a}\cdot\underline{\Sigma})\,\underline{b} = \underline{a}\wedge\underline{b}$ for $\underline{a}, \underline{b}\in\mathbb{R}^3$. Decomposing $\underline{p}$ into components parallel and orthogonal with respect to $\underline{h}$, we get

$$p_t = \mu h + \mathrm{e}^{2t\underline{h}\cdot\underline{\Sigma}}\, \underline{p}^{\perp}\cdot\underline{\sigma}. \tag{4.34}$$

A further integration and some elementary estimates lead to

$$x_t = x + t\mu h + b_t\, |h|^{-1},$$

where b_t is a bounded operator such that $\|b_t\| \leqslant 1$ for all $t\in\mathbb{R}$. Thus, for f in the dense subspace $D(x)\cap D(|h|^{-1})$, we have $\lim_{t\to+\infty} t^{-1} x_t f = \mu h f$. It follows that

$$\lim_{t\to+\infty} \frac{x_t}{t} = \mu h,$$

in the strong resolvent sense. Moreover, since μh has purely absolutely continuous spectrum, we also have

$$\operatorname*{s-lim}_{t\to+\infty} \operatorname{sign}\left(\frac{x_t}{t}\right) = \operatorname{sign}(\mu h). \quad \blacksquare$$

5. PROOFS

In this section, we complete the proofs of the results stated in Section 2. Theorem 2.1 has already been proved in the previous Section.

Proof of Theorem 2.2. By Corollary 4.4 and Lemma 4.5, the operator T_+ has purely absolutely continuous spectrum, commutes with translations and is independent of M. Using Corollary 4.2, Theorem 2.2 follows immediately from Corollary 4.10 and Lemma 4.11 in ref. 3 and the fact that

$$\sigma_3^{(x)} = \mathbf{B}(|x\rangle\langle x| \otimes \sigma_3),$$

$$S_{\mu\nu}(x, y) = \mathbf{B}(|x\rangle\langle y| \otimes s_{\mu\nu}). \quad \blacksquare$$

Proof of Theorem 2.3. From Lemma 4.5, simple manipulations lead to

$$s = \operatorname{sign} v_- = \operatorname{sign}(\mu h) = \operatorname{sign}(\mu)\,\frac{h}{|\underline{h}|}\cdot\underline{\sigma} = \operatorname{sign}(\underline{h}\cdot\underline{p})\,\frac{h}{|\underline{h}|}\cdot\underline{\sigma} = \operatorname{sign}(\kappa)\,\frac{h}{|\underline{h}|}\cdot\underline{\sigma},$$

and hence

$$sh = \operatorname{sign}(v_-)\,h = \operatorname{sign}(\kappa)\,|h|.$$

Another straightforward calculation gives

$$\operatorname{tr}(T_+) = 1 + \operatorname{sign}(\kappa)\,\frac{\operatorname{sh}\delta\,|\underline{h}|}{\operatorname{ch}\beta\,|\underline{h}| + \operatorname{ch}\delta\,|\underline{h}|},$$

$$\operatorname{tr}(\sigma_1 T_+) = 0,$$

$$\operatorname{tr}(\sigma_2 T_+) = -\frac{\operatorname{sh}\beta\,|\underline{h}|}{|\underline{h}|}\,\frac{\gamma\sin\xi}{\operatorname{ch}\beta\,|\underline{h}| + \operatorname{ch}\delta\,|\underline{h}|},$$

$$\operatorname{tr}(\sigma_3 T_+) = -\frac{\operatorname{sh}\beta\,|\underline{h}|}{|\underline{h}|}\,\frac{\lambda - \cos\xi}{\operatorname{ch}\beta\,|\underline{h}| + \operatorname{ch}\delta\,|\underline{h}|},$$

which, together with Eqs. (2.6) and (2.9) lead to Theorem 2.3. $\blacksquare$

Proof of Corollary 2.4. Since the NESS $\omega_+^{\beta_L,\beta_R}$ is a factor state, it is either normal or singular with respect to the initial state $\omega_0^{M,\beta_L,\beta_R}$ by Theorem 3.2 in ref. 16. When $\operatorname{Ep}(\omega_+^{\beta_L,\beta_R}) > 0$, the first alternative is excluded by Proposition 4.4 in ref. 16. This proves Corollary 2.4. $\blacksquare$

Proof of Theorem 2.5. Using the Jordan–Wigner transformation (3.18), the function $C_3^T(x)$ is mapped into a four-point function which can be reduced by the Wick expansion (3.26) to

$$C_3^T(x) = 4 \det \check{T}_+(x),$$

where $\check{T}_+(x)$ is the inverse Fourier transform of $T_+(\xi)$. An explicit calculation shows that

$$C_3^T(x) = -\left(\int_0^{2\pi} \frac{d\xi}{2\pi} \operatorname{sign} \kappa \, \frac{\operatorname{sh} \delta\, |h|}{\operatorname{ch} \beta\, |h| + \operatorname{ch} \delta\, |h|} \sin \xi x\right)^2 + r(x),$$

where the remainder $r(x)$ is exponentially decreasing as x tends to infinity. Theorem 2.5 follows from an elementary analysis of the singularities in the above integral. ∎

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